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# The Periodic-Dirichlet Problem for Some Semilinear Wave Equations

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DEDICATED TO THE MEMORY OF LAMBERTO CESARI

## 1. INTRODUCTION

The aim of this paper is to prove the existence and uniqueness of the solution for equations of the form

$$Lu + Nu = f, \tag{1}$$

in a Hilbert space $H$, with $L\colon \operatorname{dom} L \subset H \to H$ linear and self-adjoint, $N\colon H \to H$ a possibly nonlinear operator. First, by a direct use of the Banach contraction theorem, we are able to obtain simpler proofs and improvements of recent results of Smiley [13]. Applications are then given to the periodic-Dirichlet problem for multi-dimensional semilinear wave equations of the form

$$u_{tt} - \Delta u + g(u) = f(t, x),$$

on rectangles of $\mathbb{R}^n$ with sides commensurable with the time period.

In the one-dimensional space case with space length incommensurable to the time period, we then show the equivalence between some number theoretical assumptions introduced by McKenna [9] with other ones used earlier in [6] and we improve some existence results of [9]. In this case, we use some results of the theory of numbers obtained by Naparstek in



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[10] which are proved in a much simpler way in the Appendix. All the above mentioned authors are Ph.D. students of Lamberto Cesari whose pioneering work in the functional analytic treatment of the periodic solutions of semilinear hyperbolic equations and systems is well described in the survey papers [4, 5].

## 2. Existence and Uniqueness Results for Lipschitzian Perturbations of Self-Adjoint Operators in a Hilbert Space

Let $H$ be a real Hilbert space with inner product $(x, y)$ and corresponding norm $|x| = (x, x)^{1/2}$, $L\colon \operatorname{dom} L \subset H \to H$ a linear self-adjoint operator, $N\colon H \to H$ a (possibly) nonlinear operator and $f \in H$. We denote by $\sigma(L)$ the spectrum of $L$. For $\lambda \notin \sigma(L)$, we denote by $d_\lambda$ the distance of $\lambda$ to $\sigma(L)$. As $\sigma(L)$ is closed, $d_\lambda > 0$.

The following simple existence result, modelled on [9, Theorem 4] will be useful in the sequel.

Lemma 1. *Assume that there exists $\lambda \notin \sigma(L)$, $\mu \in [0, d_\lambda[$ and $\nu \geqslant 0$ such that the conditions*

(i) $|Nu + \lambda u - Nv - \lambda v| \leqslant d_\lambda \, |u - v|$,

(ii) $|Nu + \lambda u| \leqslant \mu \, |u| + \nu$,

*hold for all $u$, $v \in H$.*

*Then equation*

$$Lu + Nu = f, \tag{1}$$

*admits at least one solution for each $f \in H$.*

*If* (i) *and* (ii) *are replaced by*

(iii) $|Nu + \lambda u - (Nv + \lambda v)| \leqslant \mu \, |u - v|$,

*then* (1) *has a unique solution which can be obtained, from any $u_0 \in \operatorname{dom} L$, by the iteration process defined by*

$$Lu_{k+1} - \lambda u_{k+1} = f - (Nu_k + \lambda u_k), \qquad k \in \mathbb{N}.$$

*Proof.* Equation (1) is clearly equivalent to the fixed point problem in $H$

$$u = (L - \lambda I)^{-1} \, [f - (Nu + \lambda u)] = T_\lambda u$$

and, $L$ being self-adjoint,

$$|(L - \lambda I)^{-1}| = d_\lambda^{-1}.$$

Consequently, if condition (i) holds, $T_\lambda : H \to H$ is a nonexpansive operator. On the other hand, for $|u| \leqslant R$, we have, if condition (ii) holds,

$$|T_\lambda u| \leqslant d_\lambda^{-1}[\mu\, |u| + v + |f|] \leqslant d_\lambda^{-1}\mu R + d_\lambda^{-1}(v + |f|) \leqslant R,$$

provided

$$R \geqslant (1 - d_\lambda^{-1}\mu)^{-1}\, d_\lambda^{-1}(v + |f|),$$

in which case $T_\lambda : B[R] \to B[R]$, if $B[R]$ denotes the closed ball in $H$ of center 0 and radius $R$. It then follows from Browder–Göhde–Kirk fixed point theorem [15] that $T_\lambda$ has a fixed point in $B[R]$, i.e., that (1) has a solution in $\operatorname{dom} L \cap B[R]$. If condition (iii) holds instead, the same reasoning shows that $T_\lambda$ is a strict contraction on $H$ and the result follows from the Banach fixed point theorem. ∎

*Remark* 1. The first part of Lemma 1 is a slight generalization of [9, Theorem 1], which corresponds to $\lambda = 0$ and hence covers situations where $0 \notin \sigma(L)$, i.e., $L$ is invertible. A result similar to the case of assumption (iii) was considered in [8].

As an application of this Lemma 1 with $\lambda = 0$, we can consider, like in [9] the existence of weak solutions of the following periodic-Dirichlet problem for a one-dimensional semilinear wave equation

$$\begin{aligned}
u_{tt} - u_{xx} + g(u) &= f(t, x) && \text{on } ]0, \pi/\sqrt{2}[ \times ]0, \pi[ \\
u(t, 0) = u(t, \pi) &= 0 && \text{on } [0, \pi/\sqrt{2}] \qquad (2) \\
u(0, x) - u(\pi/\sqrt{2}, x) = u_t(0, x) - u_t(\pi/\sqrt{2}, x) &= 0 && \text{on } [0, \pi],
\end{aligned}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f \in L^2(]0, \pi/\sqrt{2}[ \times ]0, \pi[)$.

A *weak solution* of (2) is some $u \in L^2(]0, \pi/\sqrt{2}[ \times ]0, \pi[)$ such that

$$\int_0^{\pi/\sqrt{2}} \int_0^{\pi} [u(\phi_{tt} - \phi_{xx}) + (g(u) - f)\, \phi]\, dx\, dt = 0$$

for all $\phi \in C^2([0, \pi/\sqrt{2}] \times [0, \pi])$ such that

$$\begin{aligned}
\phi(t, 0) = \phi(t, \pi) &= 0 && \text{on } [0, \pi/\sqrt{2}], \\
\phi(0, x) - \phi(\pi/\sqrt{2}, x) = \phi_t(0, x) - \phi_t(\pi/\sqrt{2}, x) &= 0 && \text{on } [0, \pi].
\end{aligned}$$

Denoting by $L$ the abstract realization in $H = L^2(]0, \pi/\sqrt{2}[ \times ]0, \pi[)$ of the wave operator with the periodic-Dirichlet conditions on $]0, \pi/\sqrt{2}[ \times ]0, \pi[$, it is standard to show that $L$ is self-adjoint and

$$\sigma(L) = \{j^2 - 2k^2 : j \in \mathbb{N}_0, k \in \mathbb{N}\}.$$

Thus $0 \notin \sigma(L)$ and, by the theory of Pell's equation in number theory, each eigenvalue of $L$ has an infinite multiplicity [11]. As $d_0 = 1$, we shall have, by Lemma 1, existence of a weak solution of (2) for each $f \in L^2$ if

$$| g(u) - g(v)| \leqslant |u - v|,$$

for all $u$, $v \in \mathbb{R}$ and

$$| g(u)| \leqslant \mu \, |u| + \nu,$$

for some $\mu \in [0, 1[$, $\nu \geqslant 0$ and all $u \in \mathbb{R}$. For example, the assumptions are satisfied by

$$g(u) = \sin(h(u)),$$

where, for some $R > 0$ and $0 \leqslant \mu < 1, h$ is defined by $h(u) = u$ if $|u| \leqslant R$, $\mu u + (1 - \mu) R$ if $u > R$ and $\mu u - (1 - \mu) R$ if $u < -R$.

If we now write $\sigma(L) = \{\lambda_n : n \in \mathbb{Z}\}$ with $\lambda_n < \lambda_{n+1}$, a direct application of Lemma 1 with $\lambda = (\lambda_{n+1} + \lambda_n)/2$, so that $d_\lambda = (\lambda_{n+1} - \lambda_n)/2$, implies the existence of a weak solution of (2) for each $f \in L^2$ whenever

$$| g(u) - g(v) + [(\lambda_{n+1} + \lambda_n)/2](u - v)| \leqslant [(\lambda_{n+1} - \lambda_n)/2] \, |u - v|,$$

and

$$| g(u) + [(\lambda_{n+1} + \lambda_n)/2] | \leqslant \mu \, |u| + \nu,$$

for some $0 \leqslant \mu < (\lambda_{n+1} - \lambda_n)/2$, $\nu \geqslant 0$ and $u, v \in \mathbb{R}$.

Similar results hold for the case where $]0, \pi/\sqrt{2}[$ is replaced by $]0, 2\pi\sqrt{m/n}[$ for some square free positive integers $m$ and $n$.

## 3. Existence and Uniqueness Results for Strongly Monotone Perturbations of Self-Adjoint Operators in a Hilbert Space

The following consequence of Lemma 1 will cover cases where $0 \in \sigma(L)$. If $\mathbb{R}_0^-$ (resp. $\mathbb{R}_0^+$) denotes the set of negative (resp. positive) real numbers, we shall set

$$d_0^- = \operatorname{dist}(0, \sigma(L) \cap \mathbb{R}_0^-),$$

with the convention $d_0^- = +\infty$ if $\sigma(L) \backslash \{0\} \subset \mathbb{R}_0^+$. The following result generalizes in several ways [13, Theorem 3.1].

THEOREM 1. *Assume that* $0 < d_0^- < \infty$ *and that there exist positive constants* $\beta_0, \beta_1, \gamma_0, \gamma_1, \delta_0, \delta_1$ *such that the assumptions*

(1) $(Nu - Nv, u - v) \geqslant \beta_0 |u - v|^2$,

(2) $|Nu - Nv| \leqslant \beta_1 |u - v|$,

(3) $(Nu, u) \geqslant \gamma_0 |u|^2 - \delta_0$,

(4) $|Nu| \leqslant \gamma_1 |u| + \delta_1$,

*are satisfied for all* $u, v, \in H$.

*If the following conditions hold*:

(i) $\beta_1^2 \leqslant d_0^- \beta_0$,

(ii) $\gamma_1^2 < d_0^- \gamma_0$,

*then Eq.* (1) *has at least one solution for each* $f \in H$.
*If conditions* (1) *and* (2) *hold together with the inequality*

(iii) $\beta_1^2 < d_0^- \beta_0$,

*then Eq.* (1) *has, for each* $f \in H$, *a unique solution which can be obtained by the iterative process defined by* $u_0 \in \operatorname{dom} L$ *and*

$$Lu_{k+1} + (d_0^-/2)\, u_{k+1} = f - (Nu_k - (d_0^-/2)\, u_k), \qquad k \in \mathbb{N}.$$

*Proof.* For each $\lambda < 0$, we have, using conditions (1) and (2),

$$\begin{aligned} &|Nu + \lambda u - (Nv + \lambda v)|^2 \\ &\quad = |Nu - Nv|^2 + 2\lambda(Nu - Nv, u - v) + \lambda^2 |u - v|^2 \\ &\quad \leqslant (\beta_1^2 + 2\lambda\beta_0 + \lambda^2)|u - v|^2. \end{aligned} \tag{3}$$

Now, taking $\lambda = -d_0^-/2$, we have $d_\lambda = d_0^-/2$ and, by (i),

$$\beta_1^2 - d_0^- \beta_0 + (d_0^-/2)^2 \leqslant (d_0^-/2)^2 = d_\lambda^2,$$

and condition (i) of Lemma 1 holds. Similarly,

$$\begin{aligned} |Nu - (d_0^-/2)\, u|^2 &= |Nu|^2 - d_0^- (Nu, u) + (d_0^-/2)^2 |u|^2 \\ &\leqslant [\gamma_1^2 - d_0^- \gamma_0 + (d_0^-/2)^2]\, |u|^2 + 2\gamma_1\delta_1 |u| + \delta_1^2 + d_0^- \delta_0 \end{aligned}$$

and hence, by (ii), there exists $0 < \mu < d_0^-/2$ and $\nu \geqslant 0$ such that

$$|Nu - (d_0^-/2)\, u| \leqslant \mu |u| + \nu$$

for all $u \in H$, so that condition (ii) of Lemma 1 holds and the first conclusion follows. In the second case, it follows from (iii) with the same choice of $\lambda$ and from conditions (1) and (2) that

$$|Nu + \lambda u - (Nv + \lambda v)| \leqslant \mu |u - v|,$$

for some $0 < \mu < d_0^-/2$ and all $u, v, \in H$, so that condition (iii) of Lemma 1 holds and the proof is complete for $d_0^-$ finite. ▮

*Remark* 2. It follows immediately from assumptions (1) to (4) of Theorem 1 and Schwarz inequality that necessarily

$$\gamma_0 \leqslant \gamma_1, \qquad \beta_0 \leqslant \beta_1.$$

*Remark* 3. The proof of Theorem 1 is motivated by that of Zarantonello [14] in his pioneering work on monotone Lipschitzian operators.

*Remark* 4. If $d_0^- = +\infty$, then $(Lu, u) \geqslant 0$ for all $u \in \operatorname{dom} L$ and $L$ is maximal monotone (see, e.g., [2]). Then it follows from a result of Browder [3] that (1) has a solution for each $f \in H$ if $N: H \to H$ is monotone, hemi-continuous, takes bounded sets into bounded sets and is such that

$$(Nu, u)/|u| \to \infty \qquad \text{as} \quad |u| \to \infty.$$

When $N: H \to H$ is a continuous gradient operator, Theorem 1 can be replaced by the following sharper result.

THEOREM 2. *Assume that $N: H \to H$ is a continuous gradient operator, that $0 < d_0^- < \infty$ and that there exist positive constants $\beta_0, \beta_1, \gamma_1, \delta_1$ such that the assumptions*

(i) $\beta_0 |u-v|^2 \leqslant (Nu - Nv, u-v) \leqslant \beta_1 |u-v|^2$

(ii) $|Nu - (d_0^-/2) u| \leqslant \gamma_1 |u| + \delta_1$

*are satisfied for all $u, v \in H$.*
*If $d_0^-$ is finite and the following conditions hold*

(iii) $\beta_1 \leqslant d_0^-$

(iv) $\gamma_1 < d_0^-/2$

*then Eq.* (1) *has at least one solution for each $f \in H$.*
*If condition* (i) *holds together with the inequality*

$$\beta_1 < d_0^-,$$

*then Eq.* (1) *has, for each $f \in H$, a unique solution which can be obtained by the iterative process defined in Theorem* 1.

*Proof.* It follows from assumption (i) and [7, Lemma 1] that, for each $\lambda \in \mathbb{R}$,

$$|Nu + \lambda u - Nv - \lambda v| \leqslant \max(|\lambda + \beta_0|, |\lambda + \beta_1|)\, |u - v| \tag{4}$$

for all $u, v, \in H$. Taking $\lambda = -d_0^-/2$, we have $d_\lambda = d_0^-/2$ and, by condition (iii), we have

$$|\beta_1 - (d_0^-/2)| \leqslant d_0^-/2 = d_\lambda, \qquad |\beta_0 - (d_0^-/2)| \leqslant d_0^-/2 = d_\lambda,$$

so that the result follows from Lemma 1. ∎

We apply this result to a periodic-Dirichlet problem in an interval for a semilinear wave equation in $\mathbb{R}^n$. Let $\alpha_i$ $(1 \leqslant i \leqslant n)$ be positive rational numbers, $\Omega = \prod_{i=1}^n ]0, \alpha_i \pi[$, $\Delta$ the Laplacian in $\mathbb{R}^n$, $g: \mathbb{R} \to \mathbb{R}$, $J = ]0, 2\pi[$, $f \in L^2(J \times \Omega)$. We consider the existence of weak solutions for the problem

$$\begin{aligned} u_{tt} - \Delta u + g(u) &= f(t, x) && \text{in } J \times \Omega \\ u(t, x) &= 0 && \text{in } \bar{J} \times \partial\Omega \\ u(2\pi, x) - u(0, x) = u_t(2\pi, x) - u_t(0, x) &= 0 && \text{in } \bar{\Omega}, \end{aligned} \tag{5}$$

i.e., the existence of $u \in L^2(J \times \Omega)$ such that

$$\int_{J \times \Omega} [u(\phi_{tt} - \Delta\phi) + (g(u) - f)\,\phi]\,dt\,dx = 0$$

for all $\phi \in C^2(\bar{J} \times \bar{\Omega})$ such that

$$\begin{aligned} \phi(t, x) &= 0 && \text{in } \bar{J} \times \partial\Omega \\ \phi(2\pi, x) - \phi(0, x) = \phi_t(2\pi, x) - \phi_t(0, x) &= 0 && \text{in } \bar{\Omega}. \end{aligned}$$

THEOREM 3. *Let $\alpha_i = p_i/q_i$ with $p_i$ and $q_i$ positive relatively prime integers $(1 \leqslant i \leqslant n)$, and let $\|p\| = \prod_{i=1}^n p_i$. Assume that there exist*

$$0 < \beta_0 \leqslant \beta_1 \leqslant \|p\|^{-2}$$

*such that the assumption*

(i) $\beta_0 \leqslant [(g(u) - g(v))/(u - v)] \leqslant \beta_1$

*is satisfied for all $u \neq v$ in $\mathbb{R}$.*
*If, moreover, one has*

(ii) $\limsup_{|u| \to \infty} [g(u)/u] < \|p\|^{-2}$,

*then problem* (5) *has at least one weak solution.*
*If condition* (i) *holds with*

(iii) $\beta_1 < \|p\|^{-2}$,

*then problem* (5) *has a unique weak solution.*

*Proof.* Let $H = L^2(J \times \Omega)$ with the usual inner product and norm, and let $L$ be the abstract realization in $H$ of the wave operator with the periodic-Dirichlet boundary conditions in (5). Then $L$ is self-adjoint and

$$\sigma(L)=\left\{\lambda_{kl_1\cdots l_n}=\left(\frac{l_1}{\alpha_1}\right)_2+\cdots+\left(\frac{l_n}{\alpha_n}\right)^2-k^2, k\in\mathbb{N}, l_i\in\mathbb{N}_0, 1\leqslant i\leqslant n\right\}.$$

Consequently, $0\in\sigma(L)$ and $d_0^-\geqslant\|p\|^{-2}$.

If we define $N$ by $(Nu)(t,x)=g(u(t,x))$, then $N\colon H\to H$ is a continuous gradient operator and

$$\beta_0\,|u-v|^2\leqslant(Nu-Nv, u-v)\leqslant\beta_1\,|u-v|^2. \tag{6}$$

Without loss of generality (modifying $f$) we can assume that $g(0)=0$. Hence, by condition (ii), there exists $\gamma_0<\|p\|^{-2}$ and $R>0$ such that

$$0\leqslant g(u)/u\leqslant\gamma_0$$

for $|u|\geqslant R$, and, using also condition (ii)

$$\begin{aligned}-(d_0^-/2)<\beta_0-(d_0^-/2)&\leqslant[g(u)-(d_0^-/2)\,u]/u\\&\leqslant\gamma_0-(d_0^-/2)<\|p\|^{-2}-(d_0^-/2)\leqslant d_0^-/2,\end{aligned}$$

so that

$$|\,g(u)-(d_0^-/2)\,u\,|\leqslant\gamma_1\,|u|,$$

for some $0<\gamma_1<d_0^-/2$ and all $|u|\geqslant R$, which easily implies assumptions (ii) and (iv) of Theorem 2. The case of a unique solution follows directly from conditions (i), (iii), and Theorem 2. ∎

*Remark* 5. Theorem 3 improves Theorem 5.1 of Smiley in [13] which requires condition (i) with condition (iii) replaced by the stronger assumption

$$\beta_1^2<\|p\|^{-2}\beta_0.$$

## 4. The Periodic-Dirichlet Problem for Semilinear Wave Equations for Some Irrational Ratios between the Period and Interval Length

Let us consider now the existence of weak solutions for the following periodic-Dirichlet problem for a one-dimensional semilinear wave equation

$$\begin{aligned}u_{tt}-u_{xx}-g(u)&=f(t,x) &&\text{on } ]0,2\pi/\alpha[\times]0,\pi[,\\ u(t,0)=u(t,\pi)&=0 &&\text{on } [0,2\pi/\alpha],\\ u(0,x)-u(2\pi/\alpha,x)=u_t(0,x)-u_t(2\pi/\alpha,x)&=0 &&\text{on } [0,\pi],\end{aligned} \tag{7}$$

where $\alpha$ is a positive irrational number which is not the square root of an integer, $g: \mathbb{R} \to \mathbb{R}$ is continuous and $f \in L^2(]0, 2\pi/\alpha[ \times ]0, \pi[)$. A weak solution of (7) is defined as for Eq. (2) with $2\sqrt{2}$ replaced by $\alpha$, and we shall denote by $L$ the abstract realization in $H = L^2(]0, 2\pi/\alpha[ \times ]0, \pi[)$ of the wave operator with the periodic-Dirichlet conditions on $]0, 2\pi/\alpha[ \times ]0, \pi[$. Thus, $L$ is self-adjoint and its spectrum $\sigma(L)$ is the closure of the set of the eigenvalues $\{j^2 - \alpha^2 k^2 : j \in \mathbb{N}_0, k \in \mathbb{N}\}$. We refer to the Appendix for the concepts and results of number theory used in this section. We first recall a special case of [6, Theorem 1], already observed in [10], which insures that 0 does not belong to the spectrum of $L$.

LEMMA 2. *The linear periodic-Dirichlet problem*

$$\begin{aligned} u_{tt} - u_{xx} &= f(t, x) && \text{on } ]0, 2\pi/\alpha[ \times ]0, \pi[, \\ u(t, 0) = u(t, \pi) &= 0 && \text{on } [0, 2\pi/\alpha], \\ u(0, x) - u(2\pi/\alpha, x) = u_t(0, x) - u_t(2\pi/\alpha, x) &= 0 && \text{on } [0, \pi], \end{aligned}$$

*has a weak solution for each* $f \in L^2(]0, 2\pi/\alpha[ \times ]0, \pi[)$ *if and only if*

$$c_\alpha = \inf_{(m,n) \in \mathbb{Z} \times \mathbb{Z}_0} |(\alpha m)^2 - n^2| > 0,$$

*in which case one has* $\operatorname{dist}(0, \sigma(L)) = c_\alpha$ *and* $|L^{-1}| = c_\alpha^{-1}$.

THEOREM 4. *Assume that $\alpha$ has a bounded sequence of partial quotients. Then there exists $\varepsilon > 0$ such that the problem (7) has a unique weak solution for each $f \in H$ when the condition*

$$\left| \frac{g(u) - g(v)}{u - v} \right| \leqslant \varepsilon,$$

*holds for all $u, v \in \mathbb{R}$, $u \neq v$.*

*Proof.* It follows from our assumptions, Corollary of the Appendix and Lemma 2 that there exists $\varepsilon_1 = c_\alpha > 0$ such that $\sigma(L) \cap ]-\varepsilon_1, \varepsilon_1[ = \emptyset$, and if we choose any $0 < \varepsilon < \varepsilon_1$, then $\varepsilon < d_0 = c_\alpha$, so that the result follows from Lemma 1. ∎

This result was already given in [9] under the slightly more restrictive condition that $g$ is of class $C^1$ and $|g'(u)| \leqslant \varepsilon$ for all $u \in \mathbb{R}$.

We can now use a result of Amann [1], for which a simpler proof based upon Cesari's alternative method is given in [7, Corollary 1], to improve [9, Theorem 4]. We shall denote by $\sigma_{\text{ess}}(L)$ the essential spectrum of $L$.

THEOREM 5. *Assume again that $\alpha$ has a bounded sequence of partial quotients. Assume moreover that there exist real numbers $a$ and $b$ with $a \leqslant b$ such that the following conditions hold.*

(i) $[a, b] \cap \sigma_{\mathrm{ess}}(L) = \varnothing$;

(ii) $a \leqslant (g(u) - g(v))/u - v \leqslant b$ *for all* $u, v \in \mathbb{R}$, $u \neq v$;

(iii) $[\liminf_{|u| \to \infty} (g(u)/u), \limsup_{|u| \to \infty} (g(u)/u)] \cap \sigma(L) = \varnothing$.

*Then problem* (7) *has at least one weak solution for each* $f \in H$.

*Proof.* We shall show that the conditions of [7, Corollary 1] are satisfied. Assumption B in this corollary follows from conditions (i) and (ii). Letting

$$g_- = \liminf_{|u| \to \infty} \frac{g(u)}{u} \qquad \text{and} \qquad g_+ = \limsup_{|u| \to \infty} \frac{g(u)}{u},$$

it follows from condition (iii) that we can find $\lambda, \mu \in \sigma(L)$ such that $]\lambda, \mu[ \subset \rho(L)$, with $\rho(L)$ the resolvant of $L$, and

$$\lambda < g_- \leqslant g_+ < \mu.$$

Let $\beta > 0$ be such that

$$\beta < \min(\mu - g_+, g_- - \lambda).$$

Then there exists $R > 0$ such that

$$\lambda < g_- - \beta \leqslant \frac{g(u)}{u} \leqslant g_+ + \beta < \mu,$$

for all $|u| \geqslant R$, and hence

$$\left| \frac{g(u)}{u} - \frac{\lambda + \mu}{2} \right| \leqslant \min\left( g_+ + \beta - \frac{\lambda + \mu}{2}, \frac{\lambda + \mu}{2} - g_- + \beta \right)$$
$$= \gamma < \frac{\mu - \lambda}{2} = \operatorname{dist}\left( \frac{\lambda + \mu}{2}, \sigma(L) \right).$$

The conclusion follows then from [7, Corollary 1] as clearly one has $(\lambda + \mu)/2 \in [a, b] \backslash \sigma(L)$. ∎

## 5. APPENDIX: A PROBLEM IN NUMBER THEORY

The existence theorems of Section 4 require some results of number theory. Those results can essentially be found in [10] but we reproduce them here for the reader's convenience, because of the lack of availability of [10] and because our presentation is simpler.

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and let $Q_\alpha$ be the quadratic form defined on $\mathbb{Z} \times \mathbb{Z}_0$ by

$$Q_\alpha(m, n) = (\alpha m)^2 - n^2.$$

Following the discussion of [6] (which is easily adapted from the periodic–periodic case to the periodic-Dirichlet one), we want to determine a class of $\alpha$ such that

$$|Q_\alpha(m, n)| \geqslant c_\alpha > 0,$$

for some $c_\alpha > 0$ and all

$$(m, n) \in \mathbb{Z} \times \mathbb{Z}_0,$$

such that $Q_\alpha(m, n) \neq 0$. Now, $|Q_\alpha(0, n)| = n^2 \geqslant 1$ for all $n \in \mathbb{Z}_0$, and hence we can restrict ourself to the $(m, n) \in \mathbb{Z}_0 \times \mathbb{Z}_0$ such that $Q_\alpha(m, n) \neq 0$, i.e., to all $(m, n) \in \mathbb{Z}_0 \times \mathbb{Z}_0$, because, $\alpha$ being irrational, $Q_\alpha(m, n) \neq 0$ for $(m, n) \in \mathbb{Z}_0 \times \mathbb{Z}_0$. As

$$Q_\alpha(m, n) = Q_{|\alpha|}(|m|, |n|),$$

we can further assume, without loss of generality, that $\alpha > 0$ and

$$(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0.$$

Define $\Gamma_\alpha$ and $\Gamma'_\alpha$, respectively, by

$$\Gamma_\alpha = \inf_{(m,n) \neq (0,0)} |Q_\alpha(m, n)|, \qquad \Gamma'_\alpha = \liminf_{|m| + |n| \to \infty} |Q_\alpha(m, n)|.$$

Clearly, $\Gamma_\alpha \leqslant \Gamma'_\alpha$ and $\Gamma'_\alpha > 0$ if and only if $\Gamma_\alpha > 0$. Indeed, if $\Gamma'_\alpha > 0$, there exists $R > 0$ such that

$$\inf_{|m| + |n| \geqslant R} |Q_\alpha(m, n)| \geqslant \Gamma'_\alpha/2 > 0,$$

and, $\alpha$ being irrational,

$$|Q_\alpha(m, n)| = |\alpha m + n|\,|\alpha m - n| \neq 0,$$

for all $(m, n) \neq (0, 0)$, and hence has a positive lower bound on the finite set $\{(m, n) \neq (0, 0) : |m| + |n| < R\}$.

Let

$$\alpha = [a_0, a_1, ...,]$$

be the continuous fraction decomposition of $\alpha$. Recall that it is obtained as follows; put $a_0 = [\alpha]$, where $[\cdot]$ denotes the integer part. Then $\alpha = a_0 + 1/\alpha_1$ with $\alpha_1 > 1$, and we set $a_1 = [\alpha_1]$. If $a_0, a_1, ..., a_{n-1}$ and $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ are known, then $\alpha_{n-1} = a_{n-1} + 1/\alpha_n$, with $\alpha_n > 1$ and we set $a_n = [\alpha_n]$. It can be shown [11] that this process does not terminate if and only if $\alpha$ is irrational. The integers $a_0, a_1, ...,$ are the partial quotients of $\alpha$; the numbers $\alpha_1, \alpha_2, ...,$ are the complete quotients of $\alpha$ and the rationals

$$\frac{p_n}{q_n} = [a_0, a_1, ..., a_n] = a_0 + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\cdots\frac{1}{a_n},$$

with $p_n$, $q_n$ relatively prime integers, are the convergents of $\alpha$ and are such that $p_n/q_n \to \alpha$ as $n \to \infty$.

It is well known that the $p_n$, $q_n$ are recursively defined by the relations

$$p_0 = a_0,\ q_0 = 1,\ p_1 = a_0 a_1 + 1,\ q_1 = a_1,$$
$$p_n = a_n p_{n-1} + p_{n-2},\ q_n = a_n q_{n-1} + q_{n-2}.$$

The following lemma is useful for finding $\Gamma'_\alpha$.

LEMMA. *To each irrational number $\alpha$ corresponds a unique (extended) number $M(\alpha) \in [\sqrt{5}, \infty]$ having the following properties*

(i) *For each positive number $\mu < M(\alpha)$ there exist infinitely many pairs $(p_i, q_i)$ with $q_i \neq 0$, such that*

$$\left|\alpha - \frac{p_i}{q_i}\right| \leqslant \frac{1}{\mu q_i^2}.$$

(ii) *If $M(\alpha)$ is finite, then, for each $\mu > M(\alpha)$, there are only finitely many pairs $(p_i, q_i)$ satisfying the inequality*

$$\left|\alpha - \frac{p_i}{q_i}\right| \leqslant \frac{1}{\mu q_i^2}.$$

*Proof.* Let

$$\mu_i = q_i^{-2}\left|\alpha - \frac{p_i}{q_i}\right|^{-1} = q_i^{-1}\,|\alpha q_i - p_i|^{-1}, \qquad i \geqslant 1,$$
$$M(\alpha) = \limsup_{i\to\infty} \mu_i \in \mathbb{R} \cup \{+\infty\}.$$

It then follows from the elementary properties of the upper limit that $M(\alpha)$ satisfies the conditions of the lemma, with the exception of the estimate

$M(\alpha) \geqslant \sqrt{5}$. But a well-known theorem of Hurwitz [11] asserts that for infinitely many pairs $(p_i, q_i)$ one has

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{\sqrt{5}\, q_i^2},$$

so that the proof is complete. ∎

If we set

$$\mathscr{M}(\alpha) = \left\{ M \in \mathbb{R}_0^+ : \textit{infinitely many } (p_i, q_i) \textit{ satisfy } \left|\alpha - \frac{p_i}{q_i}\right| \leqslant \frac{1}{M q_i^2} \right\},$$

then the above lemma clearly states that $M(\alpha) = \sup \mathscr{M}(\alpha)$.

PROPOSITION 1. *$M(\alpha)$ is finite if and only if the sequence $(a_i)_{i \in \mathbb{N}}$ of partial quotients of $\alpha$ is bounded.*

*Proof.* We have

$$\begin{aligned} \mu_i = q_i^{-2} \left|\alpha - \frac{p_i}{q_i}\right|^{-1} &= q_i^{-2} \left|(-1)^i q_i(\alpha_{i+1} q_i + q_{i-1})\right| \\ &= \left|\alpha_{i+1} + \frac{q_{i-1}}{q_i}\right| = \left|[a_{i+1}, a_{i+2}, ...] + \frac{1}{[a_i, a_{i-1}, ..., a_1]}\right| \\ &= \left|[a_{i+1}, a_{i+2}, ...] + [0, a_i, a_{i-1}, ..., a_1]\right| \\ &= \left|[a_{i+1}] + \theta_i + \eta_i\right|, \end{aligned}$$

with $0 < \theta_i, \eta_i < 1$ for all positive integers $i$.

Thus, if $(a_i)_{i \in \mathbb{N}}$ is unbounded, one has

$$\limsup_{i \to \infty} \mu_i \geqslant \limsup_{i \to \infty} ([a_{i+1}] - 2) = +\infty,$$

and $M(\alpha) = \infty$. If $(a_i)_{i \in \mathbb{N}}$ is bounded, say, by $M$, then

$$M(\alpha) = \limsup_{i \to \infty} \mu_i \leqslant \limsup_{i \to \infty} ([a_{i+1}] + 2) < \infty. \quad ∎$$

PROPOSITION 2. *If $\alpha \in \mathbb{R}_+ \backslash \mathbb{Q}$, then*

$$\Gamma'_\alpha = 2\alpha / M(\alpha).$$

*Proof.* We have

$$|Q_\alpha(p_i, q_i)| = |\alpha q_i - p_i|\,|\alpha q_i + p_i| = \mu_i^{-1}\, |\alpha + (p_i/q_i)|,$$

and hence

$$\liminf_{i \to \infty} |Q_\alpha(p_i, q_i)| = 2\alpha/M(\alpha).$$

Now let

$$\mathcal{N}(\alpha) = \{M \in \mathbb{R}_0^+ : \text{infinitely many pairs of integers } (p, q) \text{ with } q \neq 0 \text{ satisfy } |\alpha - (p/q)| \leqslant 1/Mq^2\} \supset \mathcal{M}(\alpha).$$

It is known [11] (see also the interesting paper [12]) that if $M > 2$ and $M \in \mathcal{N}(\alpha)$, then $M \in \mathcal{M}(\alpha)$, and that, for each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\sqrt{5} \in \mathcal{M}(\alpha)$. Thus,

$$M(\alpha) = \sup \mathcal{M}(\alpha) = \sup \mathcal{N}(\alpha),$$

and hence, for $\mu > M(\alpha)$, only finitely many pairs of integers $(p, q)$ with $q \neq 0$ satisfy the inequalities

$$|Q_\alpha(p, q)| \leqslant \mu^{-1}(\alpha + (p/q)) \leqslant \mu^{-1}(2\alpha + (1/\mu q^2)),$$

which implies that

$$\Gamma'_\alpha = \liminf_{|p| + |q| \to \infty} \left[ |Q_\alpha(p, q)| - \frac{1}{\mu^2 q^2} \right] \geqslant 2\alpha/\mu.$$

Consequently, $\Gamma'_\alpha \geqslant 2\alpha/M(\alpha)$, so that the equality holds. ∎

Now, as $\Gamma'_\alpha > 0$ if and only if $\Gamma_\alpha > 0$, we also have the following

COROLLARY. *$\Gamma_\alpha > 0$ if and only if $M(\alpha) < \infty$, i.e., if and only if the sequence $(a_j)_{j \in \mathbb{N}}$ is bounded above.*

This corollary shows the equivalence between the number theoretical conditions upon $\alpha$ introduced in [6] and in [9].

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